General Motion of a Variable-Mass Flexible Rocket with Internal Flow

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This paper consists of two parts. In the first part a new and general formulation of the dynamical problems associated with the powered flight of variable-mass, flexible rockets with internal gas flow is presented. The formulation comprises six ordinary differential equations for the rigid-body motion and three partial differential equations for the elastic displacements. The equations are nonlinear and possess time-dependent coefficients. The formulation contains as special cases many of the rocket dynamics problems investigated heretofore, and should prove superior when several effects must be considered simultaneously. For a critical examination of the dynamical characteristics of variable-mass bodies, in the second part of the paper an analytical solution of the boundary-value problem with time-dependent coefficients associated with the longitudinal and transverse vibrations of an axially symmetric, variable-mass, spinning rocket is obtained. The paper scrutinizes the concept of normal-mode vibration for boost vehicles with rapid mass variation.

1. Introduction

THE behavior of a rocket in flight has been studied extensively. Research in the area of rocket dynamics has been concerned with mathematical models ranging from a rigid, variable-mass rocket to a flexible, constant-mass one, during the unpowered as well as the powered flight of the vehicle. Most of these mathematical models must be regarded as treating special aspects of a more general problem.

The treatment of a missile as a rigid body of time dependent mass has been adequately covered by many investigators, including Grubin,[1] Dryer,[2] and Leitmann.[3] The ballistic trajectories of spin- and fin-stabilized rigid bodies are treated in a book by Davis, Follin, and Blitzer.[4]

Considering the missile as a slender elastic body subjected to longitudinal acceleration, Seide[5] has treated the effect of both a compressive and a tensile force on the frequencies and mode shapes of transverse vibration. Others, such as Beal,[6] have been concerned with the problem of buckling instability of a uniform bar subjected to an end thrust as well as with the change in the body natural frequencies as a result of that thrust. A series of reports by Miles, Young, and Fowler[7] offers a comprehensive treatment of a wide range of subjects associated with the dynamics of missiles, including fuel sloshing. In all these investigations the mass variation is not accounted for.

Attempts have been made to consider simultaneously the mass variation and missile flexural elasticity by investigators such as Birnbaum[8] and Edelen.[9] Both were concerned with solid-fuel rockets and neither of them included the axial elasticity of the missile. On the other hand, Price[10] concerned himself with the internal flow in a solid-fuel rocket and ignored entirely the vehicle motion. More recently an attempt to synthesize the problem of rocket dynamics has been made by Meirovitch and Wesley.[11] This latter work accounts for the mass variation, rigid-body translation and rotation, and axial and transverse deformations but assumes the motion to be planar, which excludes spinning rockets.

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This paper can be regarded as consisting of two parts: general formulations (Secs. 2–5), and solutions (Sec. 6). The first part of this investigation represents an attempt to unify the various aspects of missile dynamics problems into one formulation. Sections 2, 3, and 4 derive general equations of motion for a flexible variable-mass rocket with internal gas flow. The motion is defined by three rigid-body translations, three rigid-body rotations, and one axial and two transverse elastic displacements. The internal gas flow effects are reduced to equivalent forces identified as the Coriolis force, the force due to the unsteadiness of the gas flow relative to the vehicle, and the reactive force. In Sec. 5 the elastic motion is specified by regarding the rocket as a bar undergoing one axial and two flexural displacements. The complete formulation reduces to six ordinary differential equations for the rigid-body motion, and three partial differential equations with the associated boundary conditions for the elastic motion. The differential equations are nonlinear and, in addition, they possess time-dependent coefficients due to the mass variation. No closed-form solution of the complete nonlinear equations can be anticipated.

As an application of the general formulation, this paper examines closely the nature of the vibrational motion of variable-mass systems as opposed to constant-mass systems. Closed-form solutions are sought as they allow for easier physical interpretation. Section 6 presents analytical solutions for the boundary-value problems with time-dependent coefficients associated with the longitudinal and transverse vibrations of an axially symmetric, variable-mass, spinning rocket in vacuum. The conclusion is that normal-mode vibration in the commonly accepted sense does not exist for variable-mass systems. The analysis should provide a check of the extent to which approximate methods based on the normal-mode concept, such as the "time-slice" method, can be used for variable-mass systems. This question may prove especially interesting when the mass variation is rapid as in the case of solid-fuel rockets.

2. Equations of Motion for a General System with Internal Flow

Let us consider a system occupying a certain volume in space at time t, namely the control volume enclosed by the control surface shown in solid line in Fig. 1. If the control volume is fixed in an inertial space, then it is shown in Ref. 12

(Sec. 5-4) that the force equation has the form

$$\mathbf{F} = \mathbf{F}_S + \mathbf{F}_B = \partial/\partial t \int_{cv} \mathbf{v}\rho dv + \int_{cs} \mathbf{v}(\rho\mathbf{v}\cdot d\mathbf{A}) \quad (1)$$

where $\mathbf{F}_S$ and $\mathbf{F}_B$ are, respectively, the resultants of the surface and body forces acting upon the system, and $\mathbf{v}$ is the flow velocity relative to the control volume.

Next we consider the case in which the control volume translates and rotates relative to an inertial space. We shall assume that part of the mass is solidly attached to the control volume, hence translating and rotating in space with it, and define a system of body axes xyz fixed with respect to the control volume, so that the force equation can be written

$$\mathbf{F}_S + \mathbf{F}_B = \int_M \mathbf{a} dM = \int_M [\mathbf{a}_0 + \dot{\mathbf{v}} + 2\boldsymbol{\omega}\mathbf{x}\mathbf{v} + \dot{\boldsymbol{\omega}}\mathbf{x}\mathbf{r} + \boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r})]dM \quad (2)$$

in which $\mathbf{a}$ is the absolute acceleration of the mass element dM, $\mathbf{a}_O$ is the acceleration of the origin O of the system xyz, $\boldsymbol{\omega}$ is the angular velocity of axes xyz, and $\mathbf{r}$ is the position of dM relative to these axes. Recognizing that if the body axes were fixed in the inertial space only the term $\int_{M_f} \dot{\mathbf{v}} dM$ would survive, where M_f is the mass moving relative to the control volume, and considering Eq. (1), we can write the force equation in the form (see Ref. 12, Sec. 5–6)

$$\mathbf{F}_S + \mathbf{F}_B = (\partial/\partial t) \int_{cv} \mathbf{v}\rho dv + \int_{cs} \mathbf{v}(\rho\mathbf{v}\cdot d\mathbf{A}) + \int_M [\mathbf{a}_0 + 2\boldsymbol{\omega}\mathbf{x}\mathbf{v} + \dot{\boldsymbol{\omega}}\mathbf{x}\mathbf{r} + \boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r})]dM \quad (3)$$

where the partial derivative $\partial/\partial t$ is to be calculated by regarding axes xyz as fixed. Next we introduce the following equivalent forces

$$\mathbf{F}_C = -2\boldsymbol{\omega}\mathbf{x} \int_{M_f} \mathbf{v} dM, \; \mathbf{F}_U = -(\partial/\partial t) \int_{M_f} \mathbf{v} dM$$
$$\mathbf{F}_R = - \int_A \mathbf{v}(\rho\mathbf{v}\cdot d\mathbf{A}) \quad (4)$$

where $\mathbf{F}_C$ is recognized as the Coriolis force, $\mathbf{F}_U$ is a force due to the unsteadiness of the relative motion, and $\mathbf{F}_R$ is referred to as a reactive force. With this notation, Eq. (3) becomes

$$\mathbf{F}_S + \mathbf{F}_B + \mathbf{F}_C + \mathbf{F}_U + \mathbf{F}_R = \int_M [\mathbf{a}_0 + \dot{\boldsymbol{\omega}}\mathbf{x}\mathbf{r} + \boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r})]dM \quad (5)$$

The terms on the right side of Eq. (5) may be regarded as pertaining to a rigid body of instantaneous mass M.

In a similar manner, the torque equation about the origin O is

$$\mathbf{N}_S + \mathbf{N}_B + \mathbf{N}_C + \mathbf{N}_U + \mathbf{N}_R = \int_M \mathbf{r}\mathbf{x}[\mathbf{a}_0 + \dot{\boldsymbol{\omega}}\mathbf{x}\mathbf{r} + \boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r})]dM \quad (6)$$

where

$$\mathbf{N}_C = -2\int_{M_f} \mathbf{r}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{v})dM$$
$$\mathbf{N}_U = -\partial/\partial t \int_{M_f} \mathbf{r}\mathbf{x}\mathbf{v} dM \quad (7)$$
$$\mathbf{N}_R = -\int_A (\mathbf{r}\mathbf{x}\mathbf{v})(\rho\mathbf{v}\cdot d\mathbf{A})$$

The significance of the various torques is self-evident. Moreover, the expression for $\mathbf{N}_U$ can be easily explained by recalling

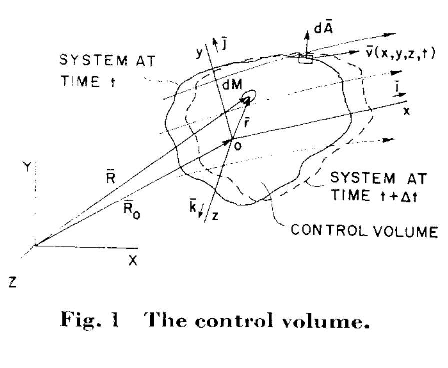

Fig. 1 The control volume.

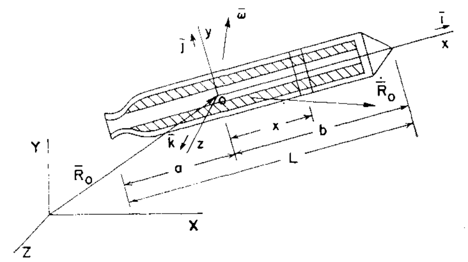

Fig. 2 Coordinate systems for the rocket in flight.

that $\partial/\partial t$ implies a time rate of change with axes xyz regarded as fixed.

These equations must be supplemented by the continuity equation

$$\int_{cs} \rho\mathbf{v}\cdot d\mathbf{A} = -\partial/\partial t \int_{cv} dM \quad (8)$$

stating that the net efflux rate of mass across the control surface equals the rate of mass decrease inside the control volume.

Recalling that the system comprises one part solid and another part of changing composition, and observing that the right sides of these equations represent the motion of the system as if it were rigid in its entirety, Eqs. (5) and (6) may be interpreted as the equations of motion of a fictitious rigid body of instantaneous mass M, provided that the actual surface and body forces acting upon the system are supplemented by the Coriolis force, the force due to the unsteadiness of the relative motion, and the reactive force. This is the substance of a statement referred to as the "principle of solidification" (see Ref. 13, p. 13).

3. Rigid-Body Motion of a Rocket

The formulation of Sec. 2 is ideally suited for problems associated with the motion of a rocket. Although a rocket is in general flexible, a first-approximation solution for its dynamic behavior may be obtained by regarding it as rigid. The solution can be refined later by considering elastic displacements. The mathematical model of the rocket is assumed to comprise a long cylindrical shell open at the aft end and closed at the fore end. The inner part of the rocket consists of the propellant which surrounds a cylindrical cavity whose axis coincides with the rocket's longitudinal axis, namely axis x in Fig. 2. The cavity plays the role of the combustion chamber, as it contains the burned gas which flows relative to the shell until expelled through a nozzle at the aft end. This mathematical model is more representative of a solid-fuel rather than liquid-fuel rocket. We shall consider first the case in which the rocket shell is rigid.

For most rockets the mass variation does not cause the vehicle center of mass to shift appreciably relative to the body (see, for example, Ref. 13, p. 15); when the fuel rate of burning and mass distribution are uniform it does not shift at all. Hence, we shall assume that the vehicle mass center is fixed relative to the body axes xyz and choose the origin O of these axes to coincide with the mass center so that, by the definition of the center of mass, we have $\int_M \mathbf{r} dM = \mathbf{0}$. As a result, Eqs. (5) and (6) reduce to

$$\mathbf{F}_S + \mathbf{F}_B + \mathbf{F}_C + \mathbf{F}_U + \mathbf{F}_R = M\mathbf{a}_0 \quad (9)$$

$$\mathbf{N}_S + \mathbf{N}_B + \mathbf{N}_C + \mathbf{N}_U + \mathbf{N}_R = \dot{\mathbf{L}}' + \boldsymbol{\omega}\mathbf{x}\mathbf{L} \quad (10)$$

where

$$\mathbf{L} = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\mathbf{i} + (-I_{xy}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\mathbf{j} + (-I_{xz}\omega_x - I_{yz}\omega_y + I_{zz}\omega_z)\mathbf{k} \quad (11)$$

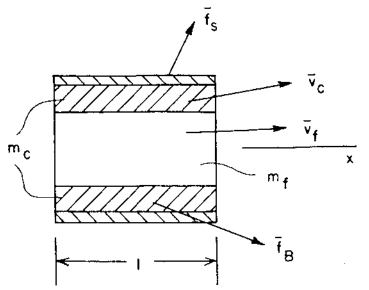

Fig. 3 The rocket element of unit length.

is the angular momentum of the vehicle about the origin O and $\dot{\mathbf{L}}'$ is the rate of change of $\mathbf{L}$ due to the change in the body angular velocity relative to the body axes. It is obtained by replacing the components of $\boldsymbol{\omega}$ by the components of $\dot{\boldsymbol{\omega}}$ in Eq. (11). The quantities

$$I_{xx} = \int_M (y^2 + z^2)dM,\ I_{yy} = \int_M (x^2 + z^2)dM$$
$$I_{zz} = \int_M (x^2 + y^2)dM,\ I_{xy} = \int_M xydM \qquad (12)$$
$$I_{xz} = \int_M xzdM,\ I_{yz} = \int_M yzdM$$

are the instantaneous moments and products of inertia of the vehicle about the body axes. Equations (9) and (10) indicate that the rigid-body translational and rotational motions are uncoupled. We note that in our case the moments of inertia are time-dependent because of the mass variation. Of course, Eq. (10) can be simplified considerably by choosing xyz to coincide with the principal axes.

It remains to derive explicit expressions for the actual and equivalent forces and torques. The surface force consists of the aerodynamic force on the vehicle wetted area and the pressure force across the exit area. Denoting by $\mathbf{f}_A{}^*$ the aerodynamic force per unit of the wetted area A_w, by p_e the pressure across the exit area A_e, and by p_a the atmospheric pressure, the surface force takes the form

$$\mathbf{F}_S = \int_{A_w} \mathbf{f}_A{}^* dA_w + (p_e - p_a)A_e\mathbf{i} \qquad (13)$$

For a uniform gravitational field, the body force is $\mathbf{F}_B = \int_L m\,\mathbf{g}dx = M\mathbf{g}$, where L is the length of the rocket, m the distributed mass, and $\mathbf{g}$ the acceleration due to gravity. Since the flow everywhere is along the x axis, with the possible exception of the exit point, we have $\mathbf{v} = -v(x,y,z,t)\mathbf{i} = -v(x,t)\mathbf{i}$, where we assumed that the flow across the cross-sectional area is uniform, so that the Coriolis force can be written

$$\mathbf{F}_C = -2\boldsymbol{\omega}\mathbf{x}\int_L \mathbf{v}m_f dx = -2(\omega_z\mathbf{j} - \omega_y\mathbf{k})\int_L \left(\int_x^b \dot{m}d\xi\right)dx \qquad (14)$$

in which use has been made of the continuity equation, namely

$$vm_f = -\int_x^b \dot{m}d\xi \qquad (15)$$

Equation (15) results from Eq. (8) by considering a control volume from a point x to the closed end of the vehicle. In Eq. (15), m_f denotes fuel mass per unit length at point x, b is the distance from the vehicle mass center to the closed end, $\dot{m}$ is the mass rate of change per unit length, and ξ is a dummy variable of integration. Similarly, the force due to the flow unsteadiness assumes the form

$$\mathbf{F}_U = -\frac{\partial}{\partial t}\int_L \mathbf{v}m_f dx = -\left[\frac{\partial}{\partial t}\int_L \left(\int_x^b \dot{m}d\xi\right)dx\right]\mathbf{i} \qquad (16)$$

Finally, the reactive force can be written

$$\mathbf{F}_R = -\int_L \left[\frac{\partial}{\partial x}(v\mathbf{v}m_f) + \Delta(v\mathbf{v}m_f)\delta(x)\right]dx = v\mathbf{v}m_f\Big|_{x_e} \qquad (17)$$

where the symbol x_e indicates that the quantity $v\mathbf{v}m_f$ is to be evaluated at the exit point. The integrand in Eq. (17) can be easily derived by assuming one-dimensional flow along the x axis. It will be noticed that the expression makes allowance for possible abrupt changes in the flow pattern, as would occur if the rocket engine were to be gimbaled at a certain angle with respect to the x direction. This is reflected by the second term in the integrand, in which $\delta(x)$ is a spatial Dirac's delta function. Letting the flow direction at the exit be defined with respect to axes xyz by the direction cosines l_{xR}, l_{yR}, l_{zR}, respectively, and using the continuity equation, Eq. (15), the reactive force becomes

$$\mathbf{F}_R = -\dot{M}v(x_e,t)(l_{xR}\mathbf{i} + l_{yR}\mathbf{j} + l_{zR}\mathbf{k}) \qquad (18)$$

where $\dot{M}$ represents the total mass rate of change, a negative quantity. The forces $\mathbf{F}_S$ and $\mathbf{F}_R$ can be replaced by $\mathbf{F}_A$ and $\mathbf{F}_T$, where $\mathbf{F}_A$ denotes the aerodynamic force

$$\mathbf{F}_A = \int_{A_w} \mathbf{f}_A{}^* dA_w \qquad (19)$$

and $\mathbf{F}_T$ is the "engine thrust"

$$\mathbf{F}_T = (p_e - p_a)A_e\mathbf{i} + |\dot{M}|v(x_e,t)(l_{xR}\mathbf{i} + l_{yR}\mathbf{j} + l_{zR}\mathbf{k}) \qquad (20)$$

In an analogous manner, we obtain the torques

$$\mathbf{N}_A = \int_{A_w} \mathbf{r}_S\mathbf{x}\mathbf{f}_A{}^* dA_w,\ \mathbf{N}_T = -a|\dot{M}|v(x_e,t)(l_{zR}\mathbf{j} - l_{yR}\mathbf{k})$$
$$\mathbf{N}_B = 0,\ \mathbf{N}_C = -2(\omega_y\mathbf{j} + \omega_z\mathbf{k})\int_L x\left(\int_x^b \dot{m}d\xi\right)dx \qquad (21)$$
$$\mathbf{N}_U = 0$$

in which $\mathbf{r}_S$ is the radius vector to a point on the rocket surface and a is the distance from the origin O to the exit point.

Equations (9) and (10), in conjunction with the expressions for the actual and equivalent forces and torques derived previously, possess time-dependent coefficients so that a closed-form solution of the problem is not possible, except for some simple special cases.

4. The Equations of Motion of a Flexible Rocket

When the rocket casing can undergo elastic deformations the problem requires further attention. The case in which the rigid-body motion is planar and the elastic motion consists of axial and transverse vibrations has been treated by Meirovitch and Wesley.[11] The present investigation represents an extension and generalization of that work.

Let us consider a rocket translating and rotating relative to the inertial space XYZ, as shown in Fig. 2. As the control volume, we consider the volume occupied by a rocket element of unit length when the vehicle is at rest relative to the body axes xyz. Figure 3 shows the corresponding element. Because the rocket shell is elastic, the entire mass associated with the control volume in question can move relative to that volume. The rocket case and unburned fuel are assumed to move together and their motion is different from the motion of the burned fuel, so that it will prove convenient to denote the motions and mass associated with the case element by the subscript c and the ones related to the (burned) fuel element by the subscript f. In analogy with Eq. (2), and considering the rocket element shown in Fig. 3, we can write the force equation of motion in the form

$$\mathbf{f}_S + \mathbf{f}_B = \int_{m_c}[\mathbf{a}_0 + \dot{\mathbf{v}}_c + 2\boldsymbol{\omega}\mathbf{x}\mathbf{v}_c + \dot{\boldsymbol{\omega}}\mathbf{x}\mathbf{r}_c + \boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r}_c)]dm +$$
$$\int_{m_f}[\mathbf{a}_0 + \dot{\mathbf{v}}_f + 2\boldsymbol{\omega}\mathbf{x}\mathbf{v}_f + \dot{\boldsymbol{\omega}}\mathbf{x}\mathbf{r}_f + \boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{r}_f)]dm \qquad (22)$$

where $\mathbf{f}_S$ and $\mathbf{f}_B$ are distributed surface and body forces, respectively, $\mathbf{v}_c$ is the elastic motion of a point inside the case element, and $\mathbf{v}_f$ is the flow velocity relative to the body axes. We shall assume that the elastic motion is the same for the entire case element, and a similar statement can be made concerning the velocity of the fuel element. Introducing the notation $\mathbf{v}_c = \dot{\mathbf{u}}$, $\mathbf{v}_f = \dot{\mathbf{u}} + \mathbf{v}$, where $\mathbf{u}$ represents the elastic displacement vector, and $\mathbf{v}$ is the velocity of the fluid relative to the case, we can rewrite Eq. (22) as follows

$$\mathbf{f}_S + \mathbf{f}_B = (\mathbf{a}_0 + \ddot{\mathbf{u}} + 2\boldsymbol{\omega}\mathbf{x}\dot{\mathbf{u}})m + \dot{\boldsymbol{\omega}}\mathbf{x}\int_m \mathbf{r}dm + \boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\int_m \mathbf{r}dm) + (\dot{\mathbf{v}} + 2\boldsymbol{\omega}\mathbf{x}\mathbf{v})m_f \quad (23)$$

where $m = m_c + m_f$ is the mass of the rocket per unit length. Moreover, the radius vector $\mathbf{r}$ has the expression $\mathbf{r} = (x + u_x)\mathbf{i} + (y + u_y)\mathbf{j} + (z + u_z)\mathbf{k}$, in which u_x, u_y, and u_z are the elastic displacements of the case element in the x,y, and z directions, respectively. A slight simplification can be achieved by assuming that the rocket possesses certain symmetry defined by $\int_m ydm = \int_m zdm = 0$, so that introducing the "average" radius vector $\bar{\mathbf{r}}$, whose definition is $\bar{\mathbf{r}} = (x + u_x)\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$, Eq. (23) assumes the form

$$\mathbf{f}_S + \mathbf{f}_B = [\mathbf{a}_0 + \ddot{\mathbf{u}} + 2\boldsymbol{\omega}\mathbf{x}\dot{\mathbf{u}} + \dot{\boldsymbol{\omega}}\mathbf{x}\bar{\mathbf{r}} + \boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\bar{\mathbf{r}})]m + (\dot{\mathbf{v}} + 2\boldsymbol{\omega}\mathbf{x}\mathbf{v})m_f \quad (24)$$

The surface forces comprise the aerodynamic forces on the surface of the rocket, as well as the forces due to stresses on the rocket shell and fluid pressure. The latter two types of forces are acting on the cross-sectional surfaces, so that, although they are internal to the rocket, they must be considered surface forces due to the nature of the chosen control volume. Body forces, as in Sec. 3, are due to gravity alone.

Invoking the analogy with Eq. (5), Eq. (24) can be rewritten as

$$\mathbf{f}_S + \mathbf{f}_B + \mathbf{f}_C + \mathbf{f}_U + \mathbf{f}_R = [\mathbf{a}_0 + \ddot{\mathbf{u}} + 2\boldsymbol{\omega}\mathbf{x}\dot{\mathbf{u}} + \dot{\boldsymbol{\omega}}\mathbf{x}\bar{\mathbf{r}} + \boldsymbol{\omega}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\bar{\mathbf{r}})]m \quad (25)$$

$$= (\mathbf{a}_0 + \bar{\mathbf{a}})m = \mathbf{a}m$$

where $\mathbf{a}$ is the absolute acceleration consisting of the acceleration $\mathbf{a}_0$ of the origin O and the acceleration $\bar{\mathbf{a}}$ of the case element relative to the body axes. Moreover

$$\mathbf{f}_C = -2\boldsymbol{\omega}\mathbf{x}\mathbf{v}m_f,\ \mathbf{f}_U = \frac{\partial}{\partial t}(\mathbf{v}m_f)$$
$$\mathbf{f}_R = -\frac{\partial}{\partial x}(v\mathbf{v}m_f) - \Delta(v\mathbf{v}m_f)\delta(x + a) \quad (26)$$

are the Coriolis force, the force due to the unsteadiness of the fluid flow relative to the case, and the reactive force, respectively, all per unit length of rocket. Recalling that the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ rotate with angular velocity $\boldsymbol{\omega}$, the absolute acceleration $\mathbf{a}$ of the case element can be shown to have the components

$$\begin{aligned}
a_x &= \ddot{R}_{0x} + \ddot{u}_x + 2[\omega_y(\dot{R}_{0z} + \dot{u}_z) - \omega_z(\dot{R}_{0y} + \dot{u}_y)] + \\
&\quad \dot{\omega}_y(R_{0z} + u_z) - \dot{\omega}_z(R_{0y} + u_y) + \omega_x\omega_y(R_{0y} + u_y) + \\
&\quad \omega_x\omega_z(R_{0z} + u_z) - (\omega_y^2 + \omega_z^2)(R_{0x} + x + u_x) \\
a_y &= \ddot{R}_{0y} + \ddot{u}_y + 2[\omega_z(\dot{R}_{0x} + \dot{u}_x) - \omega_x(\dot{R}_{0z} + \dot{u}_z)] + \\
&\quad \dot{\omega}_z(R_{0x} + x + u_x) - \dot{\omega}_x(R_{0z} + u_z) + \\
&\quad \omega_y\omega_z(R_{0z} + u_z) + \omega_x\omega_y(R_{0x} + x + u_x) - \qquad (27) \\
&\quad (\omega_x^2 + \omega_z^2)(R_{0y} + u_y) \\
a_z &= \ddot{R}_{0z} + \ddot{u}_z + 2[\omega_x(\dot{R}_{0y} + \dot{u}_y) - \omega_y(\dot{R}_{0x} + \dot{u}_x)] + \\
&\quad \dot{\omega}_x(R_{0y} + u_y) - \dot{\omega}_y(R_{0x} + x + u_x) + \\
&\quad \omega_x\omega_z(R_{0x} + x + u_x) + \omega_y\omega_z(R_{0y} + u_y) - \\
&\quad (\omega_x^2 + \omega_y^2)(R_{0z} + u_z)
\end{aligned}$$

We note that the terms involving R_{0x}, R_{0y}, R_{0z} and their time derivatives are associated with the motion of the origin O, whereas the remaining terms are due to the motion of the case element relative to O.

Similarly, using Eq. (23), the torque equation about point O for the rocket element in question takes the form

$$\mathbf{n}_S + \mathbf{n}_B = \bar{\mathbf{r}}\mathbf{x}(\mathbf{a}_0 + \ddot{\mathbf{u}} + 2\boldsymbol{\omega}\mathbf{x}\dot{\mathbf{u}})m + \dot{\mathbf{l}}' + \boldsymbol{\omega}\mathbf{x}\mathbf{l} + \bar{\mathbf{r}}\mathbf{x}(\dot{\mathbf{v}} + 2\boldsymbol{\omega}\mathbf{x}\mathbf{v})m_f \quad (28)$$

in which

$$\mathbf{l} = (i_{xx}\omega_x - i_{xy}\omega_y - i_{xz}\omega_z)\mathbf{i} + (-i_{xy}\omega_x + i_{yy}\omega_y - i_{yz}\omega_z)\mathbf{j} + (-i_{xz}\omega_x - i_{yz}\omega_y + i_{zz}\omega_z)\mathbf{k} \quad (29)$$

is the angular momentum of the mass element m about axes xyz, where

$$\begin{aligned}
i_{xx} &= \int_m [(y + u_y)^2 + (z + u_z)^2]dm \\
i_{yy} &= \int_m [(x + u_x)^2 + (z + u_z)^2]dm \\
i_{zz} &= \int_m [(x + u_x)^2 + (u + u_y)^2]dm \qquad (30) \\
i_{xy} &= \int_m (x + u_x)(y + u_y)dm,\ i_{xz} = \int_m (x + u_x) \times \\
&\quad (z + u_z)dm,\ i_{yz} = \int_m (y + u_y)(z + u_z)dm
\end{aligned}$$

are recognized as the associated moments and products of inertia. Moreover, $\dot{\mathbf{l}}'$ is obtained from Eq. (29) by replacing $\omega_x,\omega_y,\omega_z$ by $\dot{\omega}_x,\dot{\omega}_y,\dot{\omega}_z$, respectively. Equation (28) can be rewritten as

$$\mathbf{n}_S + \mathbf{n}_B + \mathbf{n}_C + \mathbf{n}_U + \mathbf{n}_R = \bar{\mathbf{r}}\mathbf{x}(\mathbf{a}_0 + \ddot{\mathbf{u}} + 2\boldsymbol{\omega}\mathbf{x}\dot{\mathbf{u}})m + \dot{\mathbf{l}}' + \boldsymbol{\omega}\mathbf{x}\mathbf{l} \quad (31)$$

where the torques

$$\mathbf{n}_C = -2\bar{\mathbf{r}}\mathbf{x}(\boldsymbol{\omega}\mathbf{x}\mathbf{v})m_f,\ \mathbf{n}_U = -\bar{\mathbf{r}}\mathbf{x}\frac{\partial}{\partial t}(\mathbf{v}m_f)$$
$$\mathbf{n}_R = -\bar{\mathbf{r}}\mathbf{x}\frac{\partial}{\partial x}(v\mathbf{v}m_f) - [\bar{\mathbf{r}}\mathbf{x}\Delta(v\mathbf{v}m_f)]\delta(x + a) \quad (32)$$

follow directly from Eqs. (26).

Equations (25) and (31) must be supplemented by the continuity equation, Eq. (15).

5. Equations for the Axial and Transverse Vibrations of a Rocket

Let us consider the rocket of the preceding section in which u_x is the axial elastic displacement and u_y and u_z are the elastic transverse displacements in the y and z directions, respectively. Assuming that the elastic displacements u_x,u_y,u_z and the angular velocity components ω_y,ω_z, as well as their time derivatives, are small quantities, we can integrate Eqs. (25) and (31) and obtain

$$\mathbf{F}_S + \mathbf{F}_B + \mathbf{F}_C + \mathbf{F}_U + \mathbf{F}_R \cong M\mathbf{a}_0 + \int_L (\ddot{\mathbf{u}} + 2\boldsymbol{\omega}\mathbf{x}\dot{\mathbf{u}})mdx \quad (33)$$

$$\mathbf{N}_S + \mathbf{N}_B + \mathbf{N}_C + \mathbf{N}_U + \mathbf{N}_R \cong \dot{\mathbf{L}}' + \boldsymbol{\omega}\mathbf{x}\mathbf{L} - \int_L x[(\ddot{u}_z + 2\omega_x\dot{u}_y + \dot{\omega}_x u_y)\mathbf{j} - (\ddot{u}_y - 2\omega_x\dot{u}_z - \dot{\omega}_x u_z)\mathbf{k}]mdx \quad (34)$$

Comparing Eqs. (9) and (33) on the one hand, and Eqs. (10) and (34) on the other hand, we conclude that the elastic motion is inertially uncoupled from the rigid body motion $\mathbf{a}_0$ and $\boldsymbol{\omega}$, provided the x axis is chosen so that the following relations are satisfied

$$\int_L \dot{\mathbf{u}}mdx = \int_L \ddot{\mathbf{u}}mdx = \int_L xu_ymdx = \int_L x\dot{u}_ymdx = \int_L x\ddot{u}_ymdx = \int_L xu_zmdx = \int_L x\dot{u}_zmdx = \int_L x\ddot{u}_zmdx = 0 \quad (35)$$

We shall assume that this is the case, and indeed Eqs. (35) imply that the elastic modes of deformation are orthogonal (with respect to mass) to the rigid body modes of displacement, namely the translation and rotation of the vehicle as a whole.

In the event the forces and torques on the vehicle do not depend on the elastic displacements the problem can be solved in two stages. In the first stage we solve for the rigid body motion $\mathbf{a}_0$ and $\boldsymbol{\omega}$ from Eqs. (9) and (10) and then, considering $\mathbf{a}_0$ and $\boldsymbol{\omega}$ as known, turn to Eqs. (25–32) for the elastic motion $\mathbf{u}$.

Equations (25) and (26), representing the equations of motion for the three components u_x,u_y,u_z of the elastic displacement $\mathbf{u}$, are of a general form and, before we can attempt their solution, we must specify the nature of the surface force $\mathbf{f}_S$. This force depends not only on the external aerodynamic forces but also on the internal stresses in the shell and gas pressure. Moreover, we must also know the flow characteristics, as can be concluded from Eq. (26).

The distributed surface force is assumed to consist of internal stresses caused by the axial and flexural vibrations (see, for example, Ref. 14, Secs. 5–7 and 10–3), internal gas pressure differential, and terms due to aerodynamic effects. The longitudinal stiffness EA_c, where E is the modulus of elasticity and A_c the cross-sectional area of the casing, and the flexural stiffnesses EI_{cy} and EI_{cz}, where I_{cy} and I_{cz} are area moments of inertia of the case about axes y and z through the cross-sectional center, take into account the casing material only. This implies that the unburned propellant possesses inertia properties but no structural stiffness.

Recalling the assumption that u_x,u_y,u_z,ω_y, and ω_z are small, the differential equation for the axial vibration takes the form

$$\frac{\partial}{\partial x}\left(EA_c\frac{\partial u_x}{\partial x}\right) - \frac{\partial}{\partial x}(pA_f) + f_{Ax} + m\mathbf{g}\cdot\mathbf{i} + \frac{\partial}{\partial t}(vm_f) - \frac{\partial}{\partial x}(v^2m_f) = m[\ddot{R}_{0x} + \ddot{u}_x + 2(\omega_y\dot{R}_{0z} - \omega_z\dot{R}_{0y}) + \dot{\omega}_yR_{0z} - \dot{\omega}_zR_{0y} + \omega_x(\omega_yR_{0y} + \omega_zR_{0z})] \quad (36)$$

where p is the gas pressure and f_{Ax} the aerodynamic force component. Equation (36) is subject to the boundary conditions

$$EA_c\frac{\partial u_x}{\partial x} = P_{x1} \text{ at } x = b,\ EA_c\frac{\partial u_x}{\partial x} = P_{x2} \text{ at } x = -a \quad (37)$$

where the functions P_{x1} and P_{x2} are axial forces exerted by the gases on the case at the ends $x = -a,b$.

In a similar manner, the differential equation for the flexural vibration in the xy-plane is

$$-\frac{\partial^2}{\partial x^2}\left(EI_{cz}\frac{\partial^2 u_y}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(P\frac{\partial u_y}{\partial x}\right) + f_{Ay} + m\mathbf{g}\cdot\mathbf{j} + 2\omega_zvm_f = m\{\ddot{R}_{0y} + \ddot{u}_y + 2[\omega_z\dot{R}_{0x} - \omega_x(\dot{R}_{0z} + \dot{u}_z)] + \dot{\omega}_z(R_{0x} + x) - \dot{\omega}_x(R_{0z} + u_z) + \omega_x\omega_y(R_{0x} + x) - \omega_x^2(R_{0y} + u_y)\} \quad (38)$$

in which $P = EA_c\partial u_x/\partial x$ denotes the axial force on the vehicle due to internal stresses. The boundary conditions are

$$\left.\begin{aligned} &EI_{cz}\frac{\partial^2 u_y}{\partial x^2} = 0,\ -\frac{\partial}{\partial x}\left(EI_{cz}\frac{\partial^2 u_y}{\partial x^2}\right) + P_{x1}\frac{\partial u_y}{\partial x} = 0 \quad \text{at } x = b \\ &EI_{cz}\frac{\partial^2 u_y}{\partial x^2} = 0,\ -\frac{\partial}{\partial x}\left(EI_{cz}\frac{\partial^2 u_y}{\partial x^2}\right) + P_{x2}\frac{\partial u_y}{\partial x} = P_{y2} \quad \text{at } x = -a \end{aligned}\right\} \quad (39)$$

The first and third of boundary conditions (39) indicate that there are no bending moments at the ends $x = -a,b$, the second one expresses the fact that the force in the y direction at the end $x = b$ is zero, and the fourth one states that there may be a transverse force P_{y2} at $x = -a$ due to a change in the flow direction at that point.

Moreover, the differential equation for the flexural vibration in the xz plane is

$$-\frac{\partial^2}{\partial x^2}\left(EI_{cy}\frac{\partial^2 u_z}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(P\frac{\partial u_z}{\partial x}\right) + f_{Az} + m\mathbf{g}\cdot\mathbf{k} - 2\omega_yvm_f = m\{\ddot{R}_{0z} + \ddot{u}_z + 2[\omega_x(\dot{R}_{0y} + \dot{u}_y) - \omega_y\dot{R}_{0x}] + \dot{\omega}_x(R_{0y} + u_y) - \dot{\omega}_y(R_{0x} + x) + \omega_x\omega_z(R_{0x} + x) - \omega_x^2(R_{0z} + u_z)\} \quad (40)$$

with the boundary conditions

$$\left.\begin{aligned} &EI_{cy}\frac{\partial^2 u_z}{\partial x^2} = 0,\ -\frac{\partial}{\partial x}\left(EI_{cy}\frac{\partial^2 u_z}{\partial x^2}\right) + P_{x1}\frac{\partial u_z}{\partial x} = 0 \quad \text{at } x = b \\ &EI_{cy}\frac{\partial^2 u_z}{\partial x^2} = 0,\ -\frac{\partial}{\partial x}\left(EI_{cy}\frac{\partial^2 u_z}{\partial x^2}\right) + P_{x2}\frac{\partial u_z}{\partial x} = P_{z2} \quad \text{at } x = -a \end{aligned}\right\} \quad (41)$$

We note that the aerodynamic forces are treated as distributed forces. Concentrated aerodynamic forces, such as at the front end of the vehicle, can be represented as distributed by means of a spatial Dirac delta function. The aerodynamic distributed forces are assumed to cause no torques on the case element. Such torques, if they exist, are assumed to affect only the rocket rigid body rotation. Although the nozzle has finite length, it was assumed, for simplicity, to be of negligible length. In fact the term P_{x2} represents the axial force on the nozzle wall from the gas flow between the two end points of the nozzle. In a more refined treatment of the gas flow, the exact pressure distribution along the finite-length nozzle may have to be considered.[12]

The flow has been treated as if it possessed no viscosity. As a result, any reactions between the gases and the unburned fuel are assumed to be normal to the flow. This is implied by the fact that the velocity is uniform over the entire cross-sectional area which implies, in turn, perfect burning in the sense that no gas-dynamic eccentricity is present. The lack of gas-dynamic eccentricity is ensured by any type of radially symmetric flow, of which the uniform flow is a special case. Any torques due to gas flow may result from engine thrust missalignment, if at all. Moreover, the velocity of the flow relative to the body is assumed to have only one component, namely along the x axis. Although, due to the transverse elastic displacements u_y and u_z, there are velocity components $v\partial u_y/\partial x$ and $v\partial u_z/\partial x$ in the y and z directions, respectively, the terms involved are assumed to be small and, therefore, ignored. Several special cases, requiring further assumptions, are discussed in the next section.

6. Axially Symmetric, Spinning Rocket in Vacuum

Our interest lies in demonstrating certain vibrational characteristics of the rocket casing during the powered flight, that is to say when the mass varies with time. Since for a solid fuel rocket the powered flight lasts only a few seconds, it is reasonable to assume that during the initial moments of the flight the aerodynamic forces are negligible. We shall further assume that the engine thrust makes a zero angle with the longitudinal axis, so that no torques are acting on the vehicle. Moreover, the mass of the rocket is considered to be uniformly distributed and to remain so during burning. Under these circumstances, the vehicle mass center will lie at the halfway point between the rocket ends, $b = a = L/2$, at all times.

We shall explore the case in which the unperturbed motion of the rocket consists of vertical upright flight in the axial direction and of spin about the longitudinal axis at the constant angular velocity ω. This motion is consistent with the assumption of no torques on the vehicle, in which case the rotational motion $\omega_x = \omega = \text{const}$, $\omega_y = \omega_z = 0$ satifies the moment equations, Eq. (10), identically. For unperturbed translational motion, two components of the force equations of motion, Eq. (9), vanish identically and only the equation for the longitudinal direction remains. Moreover, it is customary to assume that the internal gas flow is steady so that, in view of the fact that in vacuum $p_a = 0$, this component has the form

$$p_e A_e + |\dot{M}| v_e - Mg = M\ddot{R}_{0x} \tag{42}$$

where $|\dot{M}|$ is the rate of mass decrease of the entire rocket and v_e the exhaust velocity of the hot gases. But $|\dot{M}|$ is assumed to be constant in time, from which it follows that the solution of Eq. (42) has the form

$$\dot{R}_{0x} = \dot{R}_{0x}(0) + \left(\frac{p_e A_e}{|\dot{M}|} + v_e\right) \ln \frac{M_0}{M_0 - |\dot{M}|t} - gt \tag{43}$$

where M_0 is the vehicle initial mass. The remaining two velocity components are zero, $\dot{R}_{0y} = \dot{R}_{0z} = 0$. In the following we shall regard the rigid body motion as known in time.

In view of the above assumptions and results, the differential equation for the axial vibration, Eq. (36), becomes

$$\frac{\partial}{\partial x}\left(EA_c \frac{\partial u_x}{\partial x}\right) - \frac{\partial}{\partial x}(pA_f) - mg - \frac{\partial}{\partial x}(v^2 m_f) = m(\ddot{R}_{0x} + \ddot{u}_x) \tag{44}$$

which is subject to the boundary conditions

$$EA_c \frac{\partial u_x}{\partial x} = P_{x1} \text{ at } x = \frac{L}{2}, \; EA_c \frac{\partial u_x}{\partial x} = P_{x2} \text{ at } x = -\frac{L}{2} \tag{45}$$

At this point we postpone the discussion of the equations for the transverse vibration and turn our attention to the internal gas flow, which is the problem of a steady, adiabatic flow in a channel of uniform cross-sectional area. The problem is unusual in the sense that mass is continuously added to the flow at constant enthalpy and at negligible kinetic energy. An exact solution of the internal flow problem is extremely difficult and forms the subject of a separate investigation.[10] The assumption of zero viscosity implies that there are no tangential forces acting between the unburned fuel and the flowing gases so that the equation of motion for the gas alone can be separated in the form

$$-\frac{\partial}{\partial x}(pA_f) - \frac{\partial}{\partial x}(v^2 m_f) = m_f(\ddot{R}_{0x} + \ddot{u}_x + g) \tag{46}$$

which is subject to the continuity equation, Eq. (15). For uniform burning, Eq. (15) yields the relation $vm_f = m_0\beta(L - 2x)/2$, where $m_0\beta = -\dot{m} = \text{const}$ is the uniform rate of mass burning per unit length, in which $m_0 = M_0/L$ is the initial distributed mass of the rocket. It turns out that, as far as the gas flow is concerned, the right side of Eq. (46) is negligible. With this in mind, an integration of that equation yields

$$pA_f(x) = pA_f(L/2) - v^2 m_f(x) \tag{47}$$

so that the pressure drops as the gases approach the nozzle. Note that at $x = L/2$ the velocity is zero, $v(L/2) = 0$, and the pressure $p(L/2)$ is the stagnation pressure.[12]

Denoting the mass of the case and unburned fuel per unit length by $m_c = m - m_f$, regarding m_f as small compared to m_c, and introducing Eqs. (42) and (46) into Eq. (44), we obtain

$$-EA_c \frac{\partial^2 u_x}{\partial x^2} + m_c \ddot{u}_x = -\frac{1}{L}(p_e A_e + v_e M_0 \beta) \tag{48}$$

which at the end $x = L/2$ is subject to the boundary condition

$$EA_c \left.\frac{\partial u_x}{\partial x}\right|_{x=L/2} = P_{x1} = pA_f(L/2) \tag{49}$$

On the other hand, from Eq. (47), we conclude that the boundary condition at the end $x = -L/2$ is

$$EA_c \left.\frac{\partial u_x}{\partial x}\right|_{x=-L/2} = P_{x2} = pA_f(L/2) - p_e A_e - v_e M_0 \beta \tag{50}$$

To obtain the exit pressure p_e, exit velocity v_e, and mass per unit length $\rho_e A_e$ at the exit, we must analyze the compressible flow in the nozzle.[12]

Returning to the transverse vibration, we conclude that for axial symmetry, $I_{cy} = I_{cz} = I_c$, the two flexural equations of motion can be combined into a single equation by introducing the complex vector $u_{yz} = u_y + iu_z$, $i = (-1)^{1/2}$. Under the same assumptions as for the axial vibration, this definition enables us to combine Eqs. (38) and (40) into

$$-EI_c \frac{\partial^4 u_{yz}}{\partial x^4} + \frac{\partial}{\partial x}\left(P \frac{\partial u_{yz}}{\partial x}\right) = m(\ddot{u}_{yz} + 2i\omega \dot{u}_{yz} - \omega^2 u_{yz}) \tag{51}$$

whereas the boundary conditions become

$$EI_c \frac{\partial^2 u_{yz}}{\partial x^2} = 0, \; -EI_c \frac{\partial^3 u_{yz}}{\partial x^3} + P_{x1} \frac{\partial u_{yz}}{\partial x} = 0 \text{ at } x = \frac{L}{2}$$

$$EI_c \frac{\partial^2 u_{yz}}{\partial x^2} = 0, \; -EI_c \frac{\partial^3 u_{yz}}{\partial x^3} + P_{x2} \frac{\partial u_{yz}}{\partial x} = 0 \text{ at } x = -\frac{L}{2} \tag{52}$$

Note that in Eqs. (51) and (52) it was assumed that the flexural stiffness is uniform.

Examining the differential Eqs. (48) and (51), with the associated boundary conditions, we conclude that the equation for the axial elastic motion u_x can be solved independently of the equation for the transverse elastic motion u_{yz}. On the other hand, Eq. (51) depends on u_x through the axial force P so that we must solve for the axial elastic motion before attempting a solution for the transverse elastic motion.

A. Axial Vibration of a Rocket

The mathematical formulation for the axial elastic motion of the rocket comprises a nonhomogeneous differential equation, Eq. (48), to be satisfied over the entire length of the rocket, and the boundary conditions, Eqs. (49) and (50). The differential equation possesses time-dependent coefficients as the distributed mass m_c is a function of time; the axial stiffness EA_c is attributed entirely to the casing material, hence it is constant in time. In view of our assumptions concerning the relative magnitudes of the various motion components, it turns out that the axial vibration is independent not only of the transverse vibration but also of the rotation ω about the x axis.

A solution of the boundary-value problem, Eqs. (48–50), is possible by means of the modal analysis, provided the mass density m_c is constant. This, of course, is not the case but let us assume for the moment that it is. The modal analysis amounts to solving the eigenvalue problem associated with the constant-mass system, obtain the so-called normal modes, and express the system response as a superposition of the normal modes multiplied by corresponding generalized coordinates; such a solution is referred to as normal-mode vibration. Because the actual boundary-value problem possesses time-dependent coefficients, however, no normal-mode vibration is possible. Nevertheless, by virtue of the uniform-burning assumption, it turns out that a procedure based on the normal-mode approach can be used here to obtain a solution. But, because the normal modes imply a physical behavior which the actual system does not possess, we shall regard the solution as a superposition of eigenfunctions associated with the constant-mass system, rather than a superposition of normal modes.

Instead of working with a boundary-value problem consisting of a nonhomogeneous differential equation with nonhomogeneous boundary conditions, it will prove more convenient to transform the problem into another one defined by a nonhomogeneous differential equation with homogeneous boundary conditions (see Ref. 14, Sec. 7–14). To this end we introduce the transformation

$$u_x(x,t) = w(x,t) + P_1 g_1(x) + P_2 g_2(x) \tag{53}$$

where $P_1 = P_{x1}$ and $P_2 = P_{x2}$ are the same functions as in boundary conditions Eqs. (49) and (50), and g_1 and g_2 are so chosen as to render the boundary conditions in terms of $w(x,t)$ homogeneous

$$EA_c \frac{\partial w}{\partial w}\bigg|_{x=L/2} = EA_c \frac{\partial w}{\partial x}\bigg|_{x=-L/2} = 0 \tag{54}$$

It is not difficult to verify that

$$u_x(x,t) = w(x,t) + \frac{P_1}{EA_c}\left(x - \frac{L}{2}\right) h\left(x - \frac{L}{2}\right) + \frac{P_2}{EA_c}\left(x + \frac{L}{2}\right)\left[1 - h\left(x + \frac{L}{2}\right)\right] \tag{55}$$

represents the desired transformation, where $h(x - x_0)$ is a unit step function applied at $x = x_0$. Introduction of Eq. (55) into Eq. (48) yields the differential equation in terms of w

$$-EA_c \frac{\partial^2 w}{\partial x^2} + m_c \ddot{w} = P_1 \delta\left(x - \frac{L}{2}\right) - P_2 \delta\left(x + \frac{L}{2}\right) - \frac{1}{L}(p_e A_e + v_e M_0 \beta) \tag{56}$$

provided P_1 and P_2 are constant, which turns out to be the case for steady flow. Equation (56) is subject to the homogeneous boundary conditions Eq. (54).

To solve the boundary-value problem defined by Eqs. (56) and (54), we consider first the eigenvalue problem consisting of the differential equation $EA_c\phi'' + \Omega^2 m_0 \phi = 0$, over the domain $-L/2 < x < L/2$, and the boundary conditions $\phi'(L/2) = \phi'(-L/2) = 0$, where primes denote differentiations with respect to x. This eigenvalue problem corresponds to the axial vibration of a uniform, constant mass bar with both ends unrestrained. The solution of the problem can be shown to consist of the denumberably infinite set of eigenfunctions (see, for example, Ref. 14, pp. 151–154)

$$\phi_r(x) = \begin{cases} (2/m_0 L)^{1/2}(-1)^{(r+1)/2} \sin r\pi x/L, & r = 1,3,5,\ldots \\ (2/m_0 L)^{1/2}(-1)^{r/2} \cos r\pi x/L, & r = 2,4,6,\ldots \end{cases} \tag{57}$$

and the eigenvalues $\Omega_r = r\pi(EA_c/m_0L^2)^{1/2}$ $(r = 1,2,3,\ldots)$. The eigenfunctions are orthogonal to each other and, in addition, they are normalized so as to satisfy the relation

$$\int_{-L/2}^{L/2} m_0 \phi_r(x)\phi_s(x)dx = \delta_{rs} \quad (r,s = 1,2,3,\ldots)$$

where δ_{rs} is the Kronecker delta. The eigenfunction corresponding to $r = 0$ represents the rigid-body mode $\phi_0 = (1/m_0L)^{1/2}$ and the associated eigenvalue is zero, $\Omega_0 = 0$, as is to be expected for a semidefinite system. It is easy to see also that ϕ_0 is orthogonal to the eigenfunctions $\phi_s(s = 1,2,3,\ldots)$.

The solution of Eq. (56) is assumed in the form

$$w(x,t) = \sum_{r=1}^{\infty} \phi_r(x)q_r(t) \tag{58}$$

where q_r are generalized coordinates and functions ϕ_r are given by Eq. (57). Introducing Eq. (58) into Eq. (56), multiplying both sides of the result by $\phi_s(x)$, integrating over the entire domain, and recalling the properties of the functions ϕ_r, we obtain the set of uncoupled ordinary differential equations with time-dependent coefficients

$$(1 - \beta t)\ddot{q}_r + \Omega_r^2 q_r = Q_r, \quad r = 1,2,3,\ldots \tag{59}$$

where β was defined previously, and the quantities

$$Q_r = P_r\phi_r(L/2) - P_2\phi_r(-L/2), \quad r = 1,2,3,\ldots \tag{60}$$

play the role of generalized forces. Letting the initial conditions be $w(x,0) = w_0(x)$, $\dot{w}(x,0) = 0$, it is shown in Ref. 11 that the solution of Eq. (59) is

$$q_r(t) = \left[\frac{Q_r}{\Omega_r^2} + \left(w_r - \frac{Q_r}{\Omega_r^2}\right)\frac{\pi\lambda_r(1 - \beta t)^{1/2}}{2} \times \{Y_0(\lambda_r)J_1[\lambda_r(1 - \beta t)^{1/2}] - J_0(\lambda_r)Y_1[\lambda_r(1 - \beta t)^{1/2}]\}\right]$$

$$\lambda_r = \frac{2\Omega_r}{\beta}, \quad r = 1,2,3,\ldots \tag{61}$$

where

$$w_r = \int_{-L/2}^{L/2} m_0 w_0(x)\phi_r(x)dx, \quad r = 1,2,3,\ldots \tag{62}$$

and J_0, J_1, Y_0, Y_1 are Bessel functions of zero and first order and first and second kind, respectively. Solution (61) is obtained by introducing the transformation $1 - \beta t = \sigma^2$ which leads to a Bessel equation in q_r with σ as the independent variable. This completes the formal solution for the axial vibration.

B. Transverse Vibration of the Spinning Rocket

Having calculated the axial displacement u_x we are now in the position to determine the axial force $P = EA_c\partial u_x/\partial x$. This, in turn, should enable us to solve the boundary-value problem, Eqs. (51) and (52), for the transverse vibration u_{yz} of the spinning rocket. This, however, is a formidable problem because of the complicated time dependence of the coefficients introduced into Eq. (51) by P. A solution may indeed be attempted by using an eigenfunction expansion, as in the case of the axial vibration, and converting the partial differential equation into an ordinary one, whose solution may be obtained by the method of Frobenius. Practical reasons, however, render such a solution intractable. The situation is considerably improved if, through damping, the solution for the axial displacement reaches a steady-state condition in which the axial force P is no longer time dependent. We shall attempt a solution for the transverse vibration under these circumstances.

Ignoring the term containing the axial elastic acceleration $\ddot{u}_x$ in Eq. (48), and considering the boundary conditions Eqs. (49) and (50), the axial force can be shown to have the expression

$$P = EA_c\left(\frac{\partial u_x}{\partial x}\right) = P_1 - \left(\frac{1}{2} - \frac{x}{L}\right)(P_1 - P_2) \tag{63}$$

Upon introducing Eq. (63) into Eq. (51), we obtain a homogeneous partial differential equation with time-dependent coefficients entering through the mass m which is a known function of time. Equation (51) is subject to the homogeneous boundary conditions Eq. (52). A solution of the corresponding boundary-value problem can, likewise, be attempted in terms of the eigenfunctions of the associated uniform, constant-mass bar in transverse vibration but, by contrast, this time no transformation is necessary, as the boundary conditions are already in homogeneous form.

Let us consider the eigenvalue problem comprising the differential equation $EI_c\psi'''' = \Lambda^2 m_0 \psi$ and the boundary conditions $\psi'' = \psi''' = 0$ at $x = -L/2, L/2$. The solution $\psi_r(x)$, $(r = 1,2,3, \ldots)$ of this problem is given in Ref. 14, Secs. 5–10 and 10–5. Using the results from there, we can write the solution of Eq. (51), with P given by Eq. (63), in the form

$$u_{yz}(x,t) = \sum_{r=1}^{\infty} \psi_r(x)\eta_r(t) \quad (64)$$

where η_r are associated generalized coordinates, which in this case are complex. Introducing Eq. (64) into Eq. (51), multiplying the result by ψ_s, and integrating over the domain, we obtain

$$\frac{m}{m_0}(\ddot{\eta}_s + 2i\omega\dot{\eta}_s - \omega^2\eta_s) + \sum_{r=1}^{\infty} k_{rs}\eta_r = 0, \quad s = 1,2,3, \ldots \quad (65)$$

where consideration has been given to the relation $EI_c\psi_r'''' = \Lambda_r^2 m_0\psi_r$ and the fact that the eigenfunctions are orthogonal. The coefficients k_{rs} have the form

$$k_{rs} = \Lambda_r^2\delta_{rs} + \int_{-L/2}^{L/2} P\psi_r'\psi_s' dx \quad (66)$$

by virtue of the fact that u_{yz} satisfies boundary conditions (52), as can be seen in Ref. 14 (p. 447). In fact from the same source (see pp. 450–451), we can write the coefficients for $r = s$

$$\begin{aligned} k_{rr} = \Lambda_r^2 + (L/32)\{(3P_1 + P_2)[\psi_r'(L/2)]^2 + \\ (P_1 + 3P_2)[\psi_r'(-L/2)]^2\} + \\ 3[P_1\psi_r(L/2)\psi_r'(L/2) - P_2\psi_r(-L/2)\psi_r'(-L/2)] - \\ (3/L)(P_1 - P_2)[\psi_r^2(L/2) - \psi_r^2(-L/2)] \end{aligned} \quad (67)$$

whereas for $r \neq s$ we obtain

$$\begin{aligned} k_{rs} = [1/\Lambda_r^2 - \Lambda_s^2]\{\Lambda_r^2[P_1\psi_r(L/2)\psi_s'(L/2) - \\ P_2\psi_r(-L/2)\psi_s'(-L/2)] - \\ \Lambda_s^2[P_1\psi_r'(L/2)\psi_s(L/2) - P_2\psi_r'(-L/2)\psi_s(-L/2)]\} + \\ [4(P_1 - P_2)\Lambda_r^2\Lambda_s^2/L(\Lambda_r^2 - \Lambda_s^2)^2] \times \\ [\psi_r(L/2)\psi_s(L/2) - \psi_r(-L/2)\psi_s(-L/2)] \end{aligned} \quad (68)$$

Equations (65) constitute an infinite set of coupled ordinary differential equations with time-dependent coefficients; the coupling enters through the coefficients k_{rs}, $r \neq s$. But for large values of r and s the coefficients for which $r \neq s$ become increasingly small compared to the ones for which $r = s$. This is equivalent to the uncoupling of the equations corresponding to high values of r from the ones corresponding to low values of r so that we can limit the set Eq. (65) to the first n equations and truncate the series in these equations accordingly. The resulting n equations can be written in the matrix form

$$\frac{m}{m_0}\{\ddot{\eta}\} + \frac{m}{m_0}2i\omega\{\dot{\eta}\} - \frac{m}{m_0}\omega^2\{\eta\} + [k]\{\eta\} = \{0\} \quad (69)$$

where $\{\eta\}$ is an $n \times 1$ column matrix and $[k]$ is the $n \times n$ symmetric matrix of the coefficients $k_{rs}(r,s = 1,2, \ldots, n)$. It turns out that the set of n equations, Eq. (69), can be uncoupled by means of a linear transformation. To this end, we consider the eigenvalue problem associated with the symmetric matrix $[k]$ in the form

$$[k][Z] = [Z]\lceil\gamma\rfloor \quad (70)$$

where $[Z]$ is the modal matrix associated with $[k]$ and $\lceil\gamma\rfloor$ is the corresponding diagonal matrix of the eigenvalues. It can be shown (see Ref. 14, Sec. 4–8) that the modal matrix possesses the orthogonality property and, if also normalized, it satisfies the relation $[Z]^T[Z] = [I]$, where $[Z]^T$ is the transpose of $[Z]$ and $[I]$ is the identity matrix.

Next we introduce the linear transformation

$$\{\eta\} = [Z]\{\zeta\} \quad (71)$$

into Eq. (69), premultiply throughout the resulting equation by $[Z]^T$, recall that $[Z]$ is orthonormal, and obtain

$$\frac{m}{m_0}\{\ddot{\zeta}\} + \frac{m}{m_0}2i\omega\{\dot{\zeta}\} - \frac{m}{m_0}\omega^2\{\zeta\} + \lceil\gamma\rfloor\{\zeta\} = \{0\} \quad (72)$$

where, from Eqs. (70) and (71), we substituted $[Z]^T[k][Z] = \lceil\gamma\rfloor$. Equation (72) represents a set of n uncoupled equations of the form

$$(1 - \beta t)(\ddot{\zeta}_s + 2i\omega\dot{\zeta}_s - \omega^2\zeta_s) + \gamma_s\zeta_s = 0, \quad s = 1,2, \ldots, n \quad (73)$$

in which we substituted $m/m_0 \cong 1 - \beta t$ on the assumption that m_f is small relative to m_c.

Through a change of the independent variable, Eq. (73) can be brought into a form which lends itself to a closed-form solution. To show this, we introduce the transformation $\sigma^2 = 1 - \beta t$ leading to

$$\frac{d^2\zeta_s}{d\sigma^2} - \left(\frac{1}{\sigma} + \frac{4i\omega\sigma}{\beta}\right)\frac{d\zeta_s}{d\sigma} + \frac{4}{\beta^2}(\gamma_s - \omega^2\sigma^2)\zeta_s = 0, \quad s = 1,2, \ldots, n \quad (74)$$

It is not difficult to show that the general solution of Eq. (74) is

$$\zeta_s = \sigma e^{i\omega\sigma^2/\beta}[c_{s1}J_1(\alpha_s\sigma) + c_{s2}Y_1(\alpha_s\sigma)], \quad \alpha_s^2 = \frac{4\gamma_s}{\beta^2} \quad (75)$$

where the constants c_{s1} and c_{s2} depend on the initial conditions. Assuming the initial conditions

$$\zeta_s(0) = \zeta_{s0},\ d\zeta_s(t)/dt|_{t=0} = 0\ (s = 1,2, \ldots, n)$$

solution (75) becomes

$$\begin{aligned} \zeta_s(t) = \frac{\pi}{2}(1 - \beta t)^{1/2}e^{-i\omega t}\zeta_{s0}\{[2i\omega Y_1(\alpha_s) + \\ \alpha_s Y_0(\alpha_s)]J_1[\alpha_s(1 - \beta t)^{1/2}] - [2i\omega J_1(\alpha_s) + \\ \alpha_s J_0(\alpha_s)]Y_1[\alpha_s(1 - \beta t)^{1/2}]\}, \quad s = 1,2, \ldots, n \end{aligned} \quad (76)$$

We note that the initial conditions are related to the initial transverse displacement and velocity of the rocket. In particular, $d\zeta_s(t)/dt|_{t=0} = 0$ implies that the initial transverse velocity is zero. On the other hand, the quantities ζ_{s0} are related to the initial transverse displacement $y_{yz}(x,0) = u_{yz0}(x)$ by $\{\zeta_0\} = [Z]^T\{\eta_0\}$ where the sth element of the matrix $\{\eta_0\}$ has the form

$$\eta_{s0} = \eta_s(0) = \int_{-L/2}^{L/2} m_0 u_{yz0}(x)\psi_s(x)dx, \quad s = 1,2, \ldots, n \quad (77)$$

This completes the formal solution for the transverse vibration problem.

7. Numerical Results

The solution for the rigid-body motion and the one for the axial and transverse vibrations have been evaluated numerically on an IBM 7040 computer.† The data used, taken to represent a typical solid-fuel sounding rocket, is as follows: $p_e = 0$, $v_e = 8290$ ft sec^{-1}, $L = 100$ in., $m_0 g = 4.25$ lb in.$^{-1}$, $\dot{m}_c g = 0.5$, 1.57, 3.0 lb in.$^{-1}$ sec^{-1}, $E = 30 \times 10^6$ lb in.$^{-2}$, $A_c = 7.53$ in.^{2}, $I_c = 93.00$ in.^{4}, $A_{f0} = 36.4$ in.^{2}, $p_{L/2} = 2000$ lb in.$^{-2}$, $\omega = 10$ rad sec^{-1}. The initial conditions have the

† The numerical solution has been obtained by means of a computer program written by J. Bankovskis, Research Assistant, University of Cincinnati.

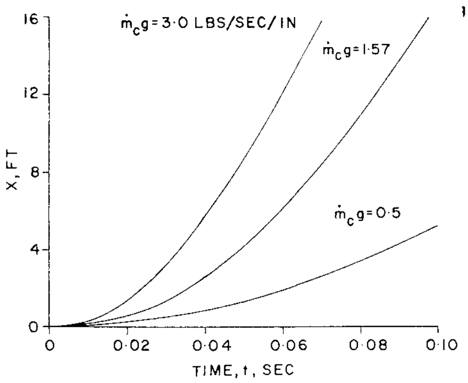

Fig. 4 Altitude vs time.

form

$$X(0) = u_x(x,0) = u_z(x,0) = 0$$

$$u_y(x,0) = \left[A\left(\cos\frac{\pi x}{L} - \frac{2}{\pi}\right) + B\left(\sin\frac{2\pi x}{L} - \frac{6x}{\pi L}\right)\right] \text{ ft}$$

where A and B are coefficients measured in feet. We note that, depending on these coefficients, $u_y(x,0)$ can be made to resemble approximately the first or the second eigenfunction of the constant-mass system. Several combinations of A and B have been explored.

Figure 4 shows the rigid-body motion in the longitudinal direction for three different burning rates. Clearly, for larger burning rates the rocket climbs faster.

Figures 5 and 6 show the axial and transverse elastic displacements for selected times. The plots are for various burning rates and a given spin velocity. We note from Figs. 5 and 6 that in the initial stages of the flight the burning rate $\dot{m}_c g$ affects the axial displacement u_x to a much larger extent than it affects the transverse displacements u_y and u_z. The explanation is that the mass rate of change has an immediate effect on the axial force and, as soon as the pressure has built up in the combustion chamber, the axial tension begins to produce an elongation of the missile case. The rate of burning affects also the transverse displacements, but this effect takes longer to make itself felt. Although the displacements u_x,u_y, and u_z are oscillatory in nature, they do not represent normal mode vibration (in the commonly accepted sense) by any means, as both the amplitude and period of oscillation depend on the burning rate and on time.

The rocket spin velocity ω has no effect whatsoever on the axial vibration but it does have an effect on the transverse vibrations. The term $e^{-i\omega t}$ represents a complex vector of unit magnitude, rotating in the negative sense with angular velocity ω. With regard to the components of Eq. (76), the term $e^{-i\omega t}[2i\omega Y_1(\alpha_s) + \alpha_s Y_0(\alpha_s)]$ can be interpreted as a

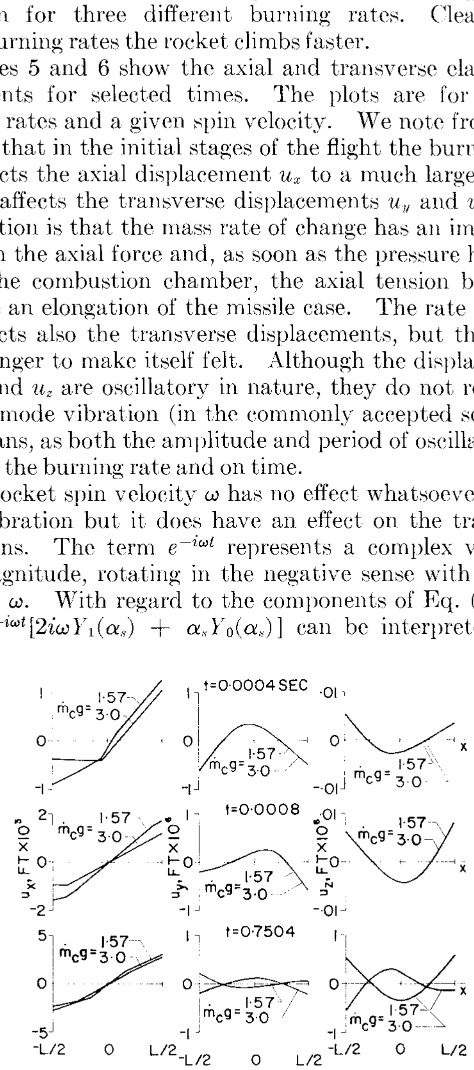

Fig. 5 Elastic displacements for $A = 0.10 \times 10^{-5}$ ft, $B = 0.25 \times 10^{-6}$ ft, and $\omega = 10$ rad/sec.

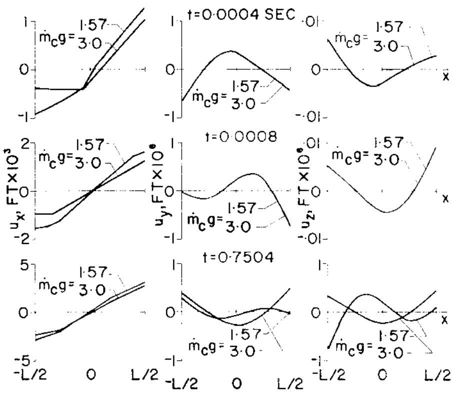

Fig. 6 Elastic displacements for $A = 0.10 \times 10^{-5}$ ft, $B = 0.50 = 10^{-6}$ ft, and $\omega = 10$ rad/sec.

rotating vector $r_1 e^{-i(\omega t - \varphi_1)}$, where $r_1 = [4\omega^2 Y_1^2(\alpha_s) + \alpha_s^2 \times Y_0^2(\alpha_s)]^{1/2}$ and $\varphi_1 = \tan^{-1}[2\omega Y_1(\alpha_s)/\alpha_s Y_0(\alpha_s)]$. A similar interpretation can be given to the term $e^{-i\omega t}[2i\omega J_1(\alpha_s) + \alpha_s J_0(\alpha_s)]$. Hence, the effect of ω is to rotate both components of the transverse vibration $\zeta_s(t)$ with the angular velocity ω with respect to the body axes but with different phase angles. Of course, the two components have different time-dependent amplitudes and periods. Since the effect of the spin rate was not found to be significant, response curves for only one value of ω are presented.

As expected, the rocket undergoes an axial displacement u_x regardless whether it was subjected to an initial displacement in the axial direction or not. By contrast, under the assumptions of zero external transverse forces and reactive transverse forces at the nozzle, the displacements u_y and u_z are entirely dependent on the transverse initial conditions. In fact, the response persists on alternating between the first and second eigenfunction of the constant-mass system. This can be easily explained by the fact that the initial transverse displacement is a combination of two functions resembling the first two eigenfunctions in question. Moreover, the assumption that the mass remains uniformly distributed throughout burning tends to eliminate the other eigenfunctions from the solutions. No tendency of the amplitudes to increase with time is detected. This is true for a relatively large range of spin rates.

8. Summary and Conclusions

In the first part of this paper (Secs. 2–5) a new and general formulation of the dynamical problems associated with the powered flight of a flexible, variable-mass rocket is presented. The unified formulation should prove superior when several effects, analyzed heretofore separately, must be treated simultaneously. This seems to be the case especially with the vehicle flexibility and mass variation, at least in the case of rapid mass variation. Indeed, to reveal the vibrational characteristics of the vehicle during the powered flight, in the second part of the paper (Sec. 6) closed-form solutions for the longitudinal and transverse vibrations under pure spin and axial rigid-body translation are obtained. These solutions show clearly that normal-mode vibration, in the commonly accepted sense, does not exist for variable-mass systems. The analytical solutions obtained here can be used to check the measure of validity of the time-slice approach, especially for systems with rapid mass variation such as the one treated here. A fact deserving special attention is that, at least for solid-fuel rockets, the engine thrust produces a tensile axial force in the missile, as a result of the internal pressure in the combustion chamber. This effect tends to reduce the transverse deformation as opposed to the unstabilizing effect of a compressive force, which would obtain if the engine thrust were assumed to be concentrated at the vehicle aft end.

It must be stressed that the formulation is of a very general nature and is applicable to a large number of problems in rocket dynamics. The two problems solved, namely the longitudinal and transverse vibrations under pure spin and axial rigid-body translation, should be regarded as special cases in which closed-form solutions are possible. No closed-form solution can be expected for the general case and a strictly numerical solution by means of a high-speed computer cannot be avoided.

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